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Complexity of Workforce Scheduling in Transfer Lines ⋆

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Abstract. Consider a production system that consists of m assembly stations arranged in series. All jobs enter the assembly line at station 1 and proceed with subsequent stations in the same order as in a flow shop. Each job spends a fixed amount of time c in each station, known as the production cycle. This production system is *synchronous* or *paced* because jobs move one station forward synchronously, every c time units. To ensure that all required work is performed in precisely c periods, the appropriate number of workers is assumed to be known for every task in each station. Hence, each job is specified by an m-tuple of workforce requirements. We are interested in "level" workforce schedules where workforce size fluctuations are minimal during the production horizon. In this article we define level workforce scheduling objectives and analyze the complexity status of the associated problems. We find that most of these problems are NP-complete even when $m = 2$.

1. Introduction

Several cells arranged in series and typically connected by a continuous material handling system form a *serial assembly line*. Such lines are designed to assemble component parts and perform any related operation necessary to produce a finished product. Group technology and serial assembly systems can be combined to produce families of parts more economically than traditional process and product layouts. In serial assembly systems the equipment to make similar parts or families of parts is grouped together in a cell designated for these parts. This way a process layout, characteristic of job shops, is changed to a small well-defined product layout. Serial assembly systems can be configured in many ways including a U shape as the Japanese have demonstrated, in an L or in a C shape, see [1].

In the article we consider problems on assembly lines consisting of several stations arranged in tandem. Each job enters the same end of the assembly line and requires a series of operations in each of the assembly stations. A distinguishing characteristic of this assembly system is that it is *paced* or *synchronous*. This means that every job spends a fixed amount of time in each station, which is the same for

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all jobs and all stations. This amount of time is called *production cycle*. Such lines are also known as *transfer lines*.

In fire truck assembly operations (that motivated this research), the necessity for a common production cycle for all assembly stations (i.e., paced assembly) comes from the size of the trucks themselves. Since it is inefficient to move semifinished trucks from station to station in real time, such movements take place off-line at the end of the 16-hour production cycle. To improve productivity, management can control the order of processing trucks, and the number of workers assigned for each truck operation. For a given production cycle of c periods, the number of workers assigned to perform a particular operation is the smallest possible that can execute the operation in c periods.

In what follows we provide a more detailed description of the application that motivates this research. A fire truck consists of three main components; the body, the chassis and the engine. The chassis and the engine are purchased from an outside supplier, while the body and final assembly (of the three main components) take place in a paced assembly line physically located in two adjacent plants. The body related operations are performed in 8 distinct stations, and the progressively assembled body is moved from station to station on a cart. A final assembly station completes the assembly line for a total of $m = 9$ stations. The physical constraints for moving semi-finished bodies from station to station dictate the common production cycle of 16 hours (i.e., $c = 16$). Workers are assigned to work on stations for 8 hours each day, and the plant runs 2 shifts. At the end of the day (i.e., the 16-hour production cycle), semifinished units are moved to the next downstream station. The 16-hour production cycle allows some flexibility in deciding how many workers are needed in each of the two daily shifts.

In this production environment workforce planning is important to maintain a competitively priced product. In this article we develop a number of workforce planning objectives and analyze the complexity status of the resulting optimization problems. The works most closely related to this article are [5], [10], and [8]. In [5] the objective is to find a job schedule that minimizes the total workforce size needed to perform a set of jobs on a serial synchronous assembly line like the one described earlier. The authors assume that every worker is trained to work in every station of the assembly line, i.e., the workforce is assumed fully cross-trained. In [10] the authors assume that each worker is only trained for a subset of stations of the assembly line. Specifically, the assembly line is partitioned and each part is referred to as a skill. Every worker of a particular skill can be assigned to any of the stations associated with that skill. The authors minimize the size of the workforce for a given partition of the assembly line into skills. In [8] the authors consider a paced job shop where jobs may visit a subset of the stations in different orders. In this protocol the length of the production schedule is no longer determined simply by the number of jobs (as in the case of a serial assembly line). The length of the schedule depends on both the workforce schedule and the workforce size. The authors minimize a linear function of the workforce size and the length of the workforce schedule. Again, it is assumed that every worker is trained to work in every station.

Note that the workforce size objective may result to schedules in which the workforce requirements from period to period vary widely. Such schedules create distractions to both the management and the workforce. From an economic viewpoint, carrying large numbers of additional workers from cycle to cycle is equivalent to carrying excess work-in-process inventory. To the best of our knowledge no research has been done on workforce planning problems on paced assembly systems with workforce leveling objectives. In fact, we are not aware of any definitions of such objectives at the day-to-day scheduling level. Workforce leveling objectives are of course known at the aggregate production level; see [11]. In this article we define a number of workforce leveling objectives and analyze the complexity of the corresponding scheduling problems. These are the main contributions of this article.

In Section 2 we present the notation and define several workforce leveling objectives and the integer programming formulations of the associated scheduling problems. In Sections 3 and 4 we consider the complexity status of the associated scheduling problems. We conclude in Section 5.

2. Description of the Assembly Line Problem

The following notation will be used throughout the paper:

n: number of jobs
m: number of production stages
c_k: the k-th production cycle
c: the time length of a production cycle; common for all production cycles
J_i: the i-th job of $J = \{J_1, J_2, \dots, J_n\}$.
ST_j: the j-th station of $ST = \{ST_1, ST_2, \dots, ST_m\}$.
W_{ij}: the workforce requirement of job J_i on station ST_j in order to complete the j-th task of J_i in c periods, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, m$.
W_k: the total (over all stations) workforce size required during cycle c_k.

We consider a serial assembly line with m stations, one station per production stage. A target production cycle of c periods is given along with a set J of n jobs. Each job $J_i \in J$ has an associated m-tuple where the j-th element W_{ij}, is the number of workers needed to process job J_i through the j-th station of the assembly line, within c time periods. All jobs enter from the first station and are transferred to the next station of the line after c periods, i.e., our assembly line is *synchronous*. Such assembly lines are often called *transfer lines*. We use mTL to denote a transfer line with m stations. The workforce requirement W_k of mTL during a production cycle c_k is the summation of the workforce requirements of all stations for cycle c_k. For this production environment, the problem of minimizing the workforce size has been considered in [5]. In this, the problem is to find a permutation of the jobs that minimizes the maximum workforce requirement over all production cycles. Let $W_{\max} = \max_k W_k$ denote this objective. Then, according to the 3-field notation in [4], the above problem is denoted by $mTL//W_{\max}$. Evidently, every permutation S of the job set J results to different workforce requirements $W_k(S)$ for each production cycle c_k and subsequently, to a different value for $W_{\max}$. Observe that the number of production cycles required to process n jobs on our synchronous assembly line is $n + m - 1$. This is because in period k, $1 \leqslant k \leqslant m - 1$, only the first k stations are busy while the last $m - k$ remain idle. The complexity status of $mTL//W_{\max}$ has been determined completely in [5] where it is shown that the problem is strongly NP-complete for $m \geqslant 3$ and solvable in $\mathcal{O}(n \log n)$ time for $m = 2$.

Objective $W_{\max}$ can be viewed as a workforce leveling objective as it minimizes the largest number of workers required over the production horizon. Since $\sum_{i,j} W_{ij}$ is constant, $W_{\max}$ should also yield level schedules. However, $W_{\max}$ does not consider explicitly the fewest number of workers used over the production horizon, nor the fluctuations in workforce size from one production cycle to another. Explicit consideration of these fluctuations offer more level worker schedules. Such workforce objectives are defined next.

2.1. LEVEL WORKFORCE PLANNING OBJECTIVES

Consider the following objective functions for workforce planning.

$W_{\min} = \min_k W_k$: the fewest number of workers required during the production horizon.

$\Delta_{k,k'} = |W_k - W_{k'}|$: represents the absolute difference in the workforce requirements of periods c_k and $c_{k'}$.

$R_{\max} = \max_{1 \leqslant k \neq k' \leqslant n+m-1} \Delta_{k,k'}$: represents the *range* of workforce requirements over the $n + m - 1$ production cycles.

$\Delta_{\max} = \max_{1 \leqslant k \leqslant n+m-2} \Delta_{k,k+1}$: represents the maximum among the absolute differences in the workforce size of consecutive production cycles.

$\sum_k \Delta_k = \sum_{k=1}^{n+m-2} \Delta_{k,k+1}$: represents the sum of the absolute differences in the workforce size of all adjacent production cycles.

By maximizing $W_{\min}$, and since $\sum_{i,j} W_{ij}$ is constant, one hopes to produce level schedules where the minimum workforce size over the production horizon is as large as possible. The $W_{\min}$ objective is opposite in nature to the $W_{\max}$ where we seek a schedule where the largest workforce size requirement over the production horizon is as small as possible. However, none of these 2 objectives considers simultaneously the largest and smallest workforce size requirements. This is done by $R_{\max}$. The range $R_{\max}$ captures fluctuations in the workforce sizes required from production cycle to production cycle by subtracting the smallest workforce requirement over the production horizon, from the maximum requirement over the same period of time. Evidently, the range objective is similar to the range statistic for a sample of data. The $\Delta_{\max}$ objective is more detailed than $R_{\max}$ in that it depends on the performance of any two adjacent production cycles. Trying to minimize $\Delta_{\max}$ implicitly affects the performance of every two adjacent cycles by placing a cap on the difference $|W_{k+1} - W_k|$. Finally, the objective $\sum_k \Delta_k$ accounts explicitly for all of the differences $|W_{k+1} - W_k|$ over the production horizon.

One can draw analogues between the above workforce objectives and the objectives used in job scheduling (see [6]). If the makespan $C_{\max}$ of a scheduling problem corresponds to the maximum workforce $W_{\max}$ of a workforce problem, then $\Delta_{\max}$ is of the same nature as the maximum tardiness $T_{\max}$, and $\sum_k \Delta_k$ is of the same nature as the total tardiness $\sum_i T_i$. In traditional scheduling, the tardiness T_i of a job is measured against a given due-date while in the workforce planning problem W_{k+1} is measured against W_k.

The workforce planning problems considered in this article for transfer assembly systems are $mTL//W_{\min}$, $mTL//R_{\max}$, $mTL//\Delta_{\max}$ and $mTL//\sum_k \Delta_k$. We also look at the cyclic versions of these problems where the problem is to identify a cyclic sequence $S = J_{[1]}J_{[2]}\ldots J_{[n]}$ (i.e., the jobs are released to the assembly line according to the sequence $J_{[1]}J_{[2]}\ldots J_{[n]}J_{[1]}J_{[2]}\ldots J_{[n]}\ldots\ J_{[1]}J_{[2]}\ldots J_{[n]}\ldots$ in a repetitive manner. The set of jobs $\{J_{[1]}, J_{[2]}, \ldots, J_{[n]}\}$ is referred to as *Minimal Product Set* (MPS). Repetitive production of MPS's is often used in environments (such as mixed-model transfer lines) that support concurrent manufacturing of a mix of products. An MPS is the smallest combination of products satisfying the demand ratios. If we have n products with demands $d_1, \ldots, d_n$, then the MPS will contain $d_1/a, \ldots, d_n/a$ units of each product (i.e., each job), where a is the greatest common divisor of the integers $d_1, \ldots, d_n$. The schedule produced for an MPS repeats itself for every MPS resulting to a smoother production of finished goods. The corresponding cyclic problems considered in this article are $mTL/cyclic/R_{\max}$, $mTL/cyclic/\Delta_{\max}$ and $mTL/cyclic/\sum_k \Delta_k$. Note that the definitions of $R_{\max}$, $\Delta_{\max}$ and $\sum_k \Delta_k$ are defined over n cycles (rather than $n+m-1$) for the case of cyclic scheduling - this is because repetitive manufacturing renders all stations busy during every production cycle.

To illustrate the different objectives as well as the cyclic version of the problems we introduce an example.

EXAMPLE 1. Consider the 2-station, 5-job problem of Table 1.

In Table 1 we compute the values $W_k(S)$ for the specific sequence $S = J_3J_4J_5J_2J_1$. In the layout used, each of the 6 columns corresponds to one of the $n + m - 1 = 6$ production cycles. For example, during the second production cycle, ST_2 requires 1 worker while ST_1 requires 8. Namely, job J_3 occupies ST_2 at the same production

Table 1. A workforce planning example for transfer lines

Job	Workforce Req's W_{i1}	W_{i2}
J_1	2	7
J_2	4	5
J_3	6	1
J_4	8	3
J_5	10	9

								k	$W_k(S)$	$\lvert W_k(S) - W_{k+1}(S)\rvert$
6	1					$\leftarrow$	J_3	1	6	3
	8	3				$\leftarrow$	J_4	2	9	4
		10	9			$\leftarrow$	J_5	3	13	0
			4	5		$\leftarrow$	J_2	4	13	6
				2	7	$\leftarrow$	J_1	5	7	0
								6	7	

cycle that J_4 occupies ST_1. For the sequence S we get that $W_{\max}(S) = 13$, and it is attained in production cycles c_3 and c_4. Also, $W_{\min} = 6$ and is attained in c_1. For the particular sequence S, observe that $R_{\max}(S) = \max_{1 \leqslant k \neq k' \leqslant 6} \Delta_{k,k'}(S) = |W_1(S) - W_3(S)| = 7$; see Table 1. Also, we see that $\Delta_{\max} = \max_k |W_k(S) - W_{k+1}(S)| = |W_4(S) - W_5(S)| = 6$ and $\sum_k \Delta_k = \sum_{k=1}^{5} \Delta_{k,k+1}(S) = 13$.

For the cyclic version of the problem, the sequence S is depicted in Table 2 together with the workforce requirements for every period.

Table 2. An example of cyclic workforce planning

									k	$W_k(S)$	$\lvert W_k(S) - W_{k+1}(S)\rvert$
6	1						$\leftarrow$	J_3	1	13	4
	8	3					$\leftarrow$	J_4	2	9	4
		10	9				$\leftarrow$	J_5	3	13	0
			4	5			$\leftarrow$	J_2	4	13	6
				2	7		$\leftarrow$	J_1	5	7	6
					6	1	$\leftarrow$	J_3	6	13	$\vdots$
						$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

In Table 2 the workforce levels for the 5 consecutive cycles starting from cycle 2 (i.e., the cycle that J_4 is released to the assembly line) are 9, 13, 13, 7, and 13.

Hence, $W_{\max}(S) = 13$, $W_{\min} = 7$, $R_{\max}(S) = 6$, $\Delta_{\max}(S) = |W_4(S) - W_5(S)| = 6$ and $\sum_k \Delta_k(S) = 20$ (note that $\sum_k \Delta_k(S)$ is calculated over a single MPS, i.e., a single processing cycle for the set J of jobs).

In what follows we provide integer programming formulations of the above defined workforce planning problems (WP) and their cyclic counterparts (CWP). In the following formulation the objective function is $W_{\max}$; all other objectives are captured with minor changes. We first introduce the decision variables

$$x_{ij} := \begin{cases} 1 & \text{if job } i \text{ is scheduled at position } [j]; \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{array}{llll}
(\mathbf{WP}) & Min \;\; W_{\max} & & \\
s.t. & \displaystyle\sum_{j=1}^{n} x_{ij} = 1 & 1 \leqslant i \leqslant n & (1) \\
& \displaystyle\sum_{i=1}^{n} x_{ij} = 1 & 1 \leqslant j \leqslant n & (2) \\
& \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{k} W_{ij} x_{i,k-j+1} \leqslant W_{\max} & 1 \leqslant k \leqslant m-1 & (3) \\
& \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} W_{ij} x_{i,k-j+1} \leqslant W_{\max} & m \leqslant k \leqslant n & (4) \\
& \displaystyle\sum_{i=1}^{n}\sum_{j=k-n+1}^{m} W_{ij} x_{i,k-j+1} \leqslant W_{\max} & n+1 \leqslant k \leqslant n+m-1 & (5) \\
& x_{ij} \in \{0, 1\} & & (6)
\end{array}$$

Equations (1) and (2) correspond to the assignment constraints, and inequalities (3)-(5) correspond to production cycles $1, 2, \ldots, n+m-1$. In particular, inequalities (3) correspond to production cycles $1, 2, \ldots, m-1$, (4) to production cycles $m, m+1, \ldots, n$ and (5) to production cycles $n+1, \ldots, n+m-1$. The above formulation can be modified for level workforce schedules by observing that

$$W_k = \begin{cases} \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{k} W_{ij} x_{i,k-j+1} & \text{if } k = 1, 2, \ldots, m-1 \\ \displaystyle\sum_{i=1}^{n}\sum_{j=1}^{m} W_{ij} x_{i,k-j+1} & \text{if } k = m, m+1, \ldots, n \\ \displaystyle\sum_{i=1}^{n}\sum_{j=k-n+1}^{m} W_{ij} x_{i,k-j+1} & \text{if } k = n+1, n+2, \ldots, n+m-1. \end{cases}$$

The cyclic version of (WP) for the $W_{\max}$ objective is given below.

$$
\begin{aligned}
(\textbf{CWP}) \quad & MIN && W_{\max} \\
& s.t. && (1), (2), (6) \\
& && \sum_{i=1}^{n}\sum_{j=1}^{m} W_{ij} x_{i,(k-j+1) mod n} \leqslant W_{\max} \quad k = 1, \ldots, n \qquad (7)
\end{aligned}
$$

where mod denotes the modulus operation and we define $n \; mod \; n = n$. In CWP the same operations are repeated every n production cycles and in each of these cycles all stations of the assembly line are busy. The workforce requirements for the production cycle c_k are captured by the set (7) of constraints. In CWP we have that $W_k = \sum_{i=1}^{n} \sum_{j=1}^{m} W_{ij} x_{i,(k-j+1) mod n}$ for $k = 1, 2, \ldots, n$. With this observation, one can capture any of the workforce leveling objectives $W_{\min}$, $R_{\max}$, $\Delta_{\max}$ and $\sum_k \Delta_k$ with minor changes.

In the rest of the paper we settle the complexity of leveling problems for the objectives $R_{\max}$, $\Delta_{\max}$ and $\sum_k \Delta_k$ while the complexity of problem $2TL//W_{\min}$ has been recently resolved in [9] where an $\mathcal{O}(n \log n)$ optimal algorithm is presented. In the following table we indicate the complexity of each problem and cite the corresponding reference. We use the mark ! to indicate that the corresponding problem is strongly NP-complete.

3. Problems $\mathbf{2TL//\Delta_{max}}$ and $\mathbf{2TL//\sum_k \Delta_k}$ are NP-complete

In this section we settle the complexity status of problems $2TL//\Delta_{\max}$ and $2TL//\sum_k \Delta_k$. We use reductions from the Numerical 3-Dimensional Matching problem which is known to be $\mathcal{NP}$-complete in the strong sense; see [2].

Numerical 3-Dimensional Matching (N3DM)

Instance: Given disjoint sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ each containing n nonnegative integers, and a bound $\bar{W}$.

Question: Can $\mathcal{A} \cup \mathcal{B} \cup \mathcal{D}$ be partitioned into n disjoint sets $S_1, S_2, \ldots, S_n$ such that each S_i contains exactly one element from each of $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and such that $\sum_{s \in S_i} s = \bar{W}$ for $i = 1, 2, \ldots, n$?

In the following theorem we show that all of our workforce planning problems on mTL's are strongly $\mathcal{NP}$-complete for $m \geqslant 3$.

THEOREM 1. *The problems $3TL//f$ are $\mathcal{NP}$-complete in the strong sense for every $f \in \{W_{max}, W_{\min}, R_{\max}, \Delta_{\max}, \sum_k \Delta_k\}$.*

Proof. Let $X = (\mathcal{A}, \mathcal{B}, \mathcal{D})$ be an instance of N3DM where $\mathcal{A} = \{a_i\}_{i=1}^n$, $\mathcal{B} = \{b_i\}_{i=1}^n$, $\mathcal{D} = \{d_i\}_{i=1}^n$. From X, construct an instance I of $3TL$ having the following $5n + 1$ jobs:

Table 3. Complexity of $mTL//f$

f :	$W_{\max}$	$W_{\min}$	$R_{\max}$	$\Delta_{\max}$	$\sum_k \Delta_k$
$m = 3$	!, [5]	!, Theorem 1	!, Theorem 1	!, Theorem 1	!, Theorem 1
$m = 2$	$O(n \log n)$, [5]	$O(n \log n)$,[9]	!, Theorem 4	!, Theorem 2	!, Theorem 3

$\{(\bar{W}, 1, a) : a \in \mathcal{A}\}$
$\{(\bar{W} - 1, \bar{W} - a, a) : a \in \mathcal{A}\}$
$\{(0, b, 0) : b \in \mathcal{B}\}$
$\{(d, d, 0) : d \in \mathcal{D}\}$
$\{(\bar{W} - d, 0, 0) : d \in \mathcal{D}\}$
$\{(\bar{W}, \bar{W}, \bar{W})\}$

where $\bar{W} = \frac{1}{n}\sum_i (a_i + b_i + d_i)$. For this instance it is shown in [5] that there exists a solution for X, if and only if I has a solution where the workforce requirement $\bar{W}_k$ equals $\bar{W}$ for every production cycle $c_1, c_2, \ldots, c_{n+2}$.

Hence, there exists a solution for X, if and only if there exists a solution for I where $W_{\max} = \bar{W}$, $W_{\min} = \bar{W}$, $R_{\max} = 0$, $\Delta_{\max} = 0$, and $\sum_k \Delta_{\max} = 0$. This means that $3TL//f$ is strongly $\mathcal{NP}$-complete for every $f \in \{W_{\max}, R_{\max}, \Delta_{\max}, \sum_k \Delta_k\}$. This completes the proof of the theorem. □

If in the above instance I, job $(\bar{W}, \bar{W}, \bar{W})$ is replaced by job $(\bar{W}, 0, 0)$, then the same proof as in [5] shows that there exists a cyclic schedule of the $5n + 1$ jobs where $W_k = \bar{W}$ for every $k = 1, 2, \ldots, 5n + 1$ if and only if there exists a solution for the instance X of N3DM. Hence, we have the following corollary.

COROLLARY 1. *The problems $3TL/cyclic/f$ are $\mathcal{NP}$-complete in the strong sense for every $f \in \{W_{\max}, W_{\min}, R_{\max}, \Delta_{\max}, \sum_k \Delta_k\}$.*

Due to Theorem 1, for a complete complexity characterization of workforce planning problems we only need to examine the 2-station cases. Recall that, in [5], the authors have presented an $\mathcal{O}(n \log n)$ algorithm for $2TL//W_{\max}$. In [3], the authors extended this algorithm for the $2TL/cyclic/W_{\max}$ case. Hence, we are only left with the objectives $\Delta_{\max}$, $\sum_k \Delta_k$, $W_{\min}$ and $R_{\max}$. In Theorems 2 and 3 we show that the 2-station workforce planning problem for the first two objectives is strongly NP-complete.

THEOREM 2. *The problem $2TL//\Delta_{\max}$ is $\mathcal{NP}$-complete in the strong sense.*

Proof. Let $X = (\mathcal{A}, \mathcal{B}, \mathcal{D})$ be an instance of N3DM. From X, construct an instance I of $2TL//\Delta_{\max}$ having the following $3n + 1$ jobs:

$(3B, 2B + a_i)$, for every $a_i \in \mathcal{A}$, $i = 1, 2, \ldots, n$, where $B = 5\bar{W}$ and $\bar{W} = \frac{1}{n}\sum_i (a_i + b_i + d_i)$,

$(B + b_i, B + \bar{W})$, for every $b_i \in \mathcal{B}$, $i = 1, 2, \ldots, n$,
$(2B + 2\bar{W} - d_i, \bar{W} - d_i)$, for every $d_i \in \mathcal{D}$, $i = 1, 2, \ldots, n$,
the single job $(3B, 3B)$.

We will show that there exists a solution for the instance X of N3DM if and only if there exists a solution for the instance I of $2TL//\Delta_{\max}$ such that $\Delta_{\max} = 2\bar{W}$.

Suppose that there exists a solution of X such that $a_i + b_i + d_i = \bar{W}$ for every $i = 1, 2, \ldots, n$. Then, for each triplet (a_i, b_i, d_i) form the 3-job block indicated below.

$$\begin{array}{llllll} 3B & 2B + a_i & & & & \\ & B + b_i & & B + \bar{W} & & \\ & & & 2B + 2\bar{W} - d_i & & \bar{W} - d_i \end{array}$$

Let S be the schedule formed by concatenating the n blocks corresponding to the triplets (a_i, b_i, d_i) (in any order) with the last job being $(3B, 3B)$. Without loss of generality assume that the n blocks are concatenated in the order $1, 2, \ldots, n$ in S. Then, $\Delta_{1,2} = \bar{W} - d_1$, $\Delta_{2+3k,3+3k} = 2\bar{W}$ for $k = 0, 1, \ldots, n-1$, $\Delta_{3k,1+3k} = 2\bar{W}$ for $k = 1, \ldots, n-1$, $\Delta_{1+3k,2+3k} = |d_{k+1} - d_k|$ for $k = 1, \ldots, n-1$, and $\Delta_{3n+1,3n+2} = \bar{W} - d_n$. Hence, in S we have that $\Delta_{\max} = 2\bar{W}$. This means that a solution for X induces a solution for I.

To show the opposite, let S be a solution for $2TL//\Delta_{\max} = 2\bar{W}$. We will first show that in S, $W_k \geqslant 3B$ for every cycle c_k, $k = 1, 2, \ldots, 3n+2$. Indeed, note that

$$\sum_{k=1}^{3n+2} W_k = (9n+6)B + 2n\bar{W} + 3\sum_{i=1}^{n}(a_i + b_i) > (9n+6)B.$$

If for some production cycle we have $W_k < 3B$, there are three ways that this can happen. Either S includes two adjacent jobs of the form $(B + b_i, B + \bar{W})$ and $(B+b_j, B+\bar{W})$ (in this case we have $W_k = 2B+\bar{W}+b_j$ during some cycle c_k), or S includes the adjacent jobs $(2B+2\bar{W}-d_i, \bar{W}-d_i)$ and $(2B+2\bar{W}-d_j, \bar{W}-d_j)$ (in this case we have $W_k = 2B + 3\bar{W} - d_i - d_j$ during some cycle c_k), or S includes the adjacent jobs $(2B + 2\bar{W} - d_i, \bar{W} - d_i)$ and $(B + b_j, B + \bar{W})$ (in this case we have $W_k = B + \bar{W} - d_i + b_j$ during some cycle c_k). In all cases we have $W_k < 2B + 3\bar{W} < 3B$. Since $\sum_k W_k > (3n+2)3B$, there must exist adjacent production cycles c_k and c_{k+1} in S (among the $3n+2$ possible ones) such that $\Delta_{k,k+1} > 3B - (2B + 3\bar{W}) = B - 3\bar{W} = 2\bar{W}$; contradicting the fact that in S, $\Delta_{\max} = 2\bar{W}$. Thus, we have to assume that in S, $W_k \geqslant 3B$ for every $k = 1, 2, \ldots, 3n+2$. With this observation, and the fact that all the parameters a_i, b_i, d_i, and $2\bar{W}$ are significantly smaller than B, we conclude that S must be formed by concatenating n blocks of the form depicted below, each consisting of three jobs, with job $(3B, 3B)$ being last.

$$\begin{array}{llll} 3B & 2B+a_i & & \\ & B+b_j & B+\bar{W} & \\ & & 2B+2\bar{W}-d_l & \bar{W}-d_l \end{array}$$

Without loss of generality, assume that the block depicted above is the k-th in sequence. Then, $\Delta_{1+3k,2+3k} = a_i + b_j < 2\bar{W}$ for $k = 0, 2, \ldots, n-1$, $\Delta_{2+3k,3+3k} = 3\bar{W} - a_i - b_j - d_l$ for $k = 0, 2, \ldots, n-1$, $\Delta_{3k,1+3k} = 2\bar{W}$ for $k = 1, 2, \ldots, n$, and $\Delta_{3n+1,3n+2} = \bar{W} - d_l < 2\bar{W}$. Since $\Delta_{\max} = 2\bar{W}$, we must have that $\Delta_{2+3k,3+3k} = 3\bar{W} - a_i - b_j - d_l \leqslant 2\bar{W}$ for $i = 1, 2, \ldots, n$, or equivalently, $\bar{W} \leqslant a_i + b_j + d_l$ for every one of the n 3-job blocks in S. If we add up the latter inequalities for $i = 1, 2, \ldots, n$ we get

$$n\bar{W} \leqslant \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i + \sum_{i=1}^{n} d_i.$$

By definition of X, we know that the above inequality holds as an equality which means that in S we must have $\bar{W} = a_i + b_j + d_l$ for every one of the n blocks. This means that S induces a solution for the instance X of N3DM. This completes the proof of the theorem. □

COROLLARY 2. *The problem $2TL/cyclic/\Delta_{\max}$ is $\mathcal{NP}$-complete in the strong sense.*

Proof. The instance I used in this proof is the same as in Theorem 2 except that job $(3B, 3B)$ is replaced by $(3B, 0)$. With this change, the complexity proof of Theorem 2 carries through for the cyclic case; the zero operation of $(3B, 0)$ ensures that $W_1 = 3B$ in the cyclic schedule. This completes the proof of the corollary. □

In the next theorem we consider the objective $\sum_k \Delta_k$.

THEOREM 3. *The problem $2TL//\sum_k \Delta_k$ is $\mathcal{NP}$-complete in the strong sense.*

Proof. Let $X = (\mathcal{A}, \mathcal{B}, \mathcal{D})$ be an instance of N3DM. From X, construct an instance I of $2TL//\sum_k \Delta_k$ having the following $8n+1$ jobs:

$(24T, 3T + a_i)$, for every $a_i \in \mathcal{A}$, $i = 1, 2, \ldots, n$, where $T = 5n\bar{W}$ and $\bar{W} = \frac{1}{n}\sum_i (a_i + b_i + d_i)$,
$(21T, 6T + a_i)$, for $i = 1, 2, \ldots, n$,
$(18T + b_i, 9T)$, for every $b_i \in \mathcal{B}$, $i = 1, 2, \ldots, n$,
$(15T + \bar{W} - d_i, 12T + \bar{W})$, for every $d_i \in \mathcal{D}$, $i = 1, 2, \ldots, n$,
$(12T, 15T + \bar{W} - d_i)$, for $i = 1, 2, \ldots, n$,
$(9T, 18T + b_i)$ for $i = 1, 2, \ldots, n$,
$(6T + a_i, 21T + a_i)$ for $i = 1, 2, \ldots, n$,
n copies of the job $(3T, 0)$ and the single job $(24T, 24T)$.

We will show that there exists a solution for the instance X of N3DM if and only if there exists a solution for the instance I of $2TL//\sum_k \Delta_k$ such that $\sum_k \Delta_{k,k+1} = 2n\bar{W}$. In the proof that follows we assume that $\max_i a_i < \min_i d_i$ and $\max_i a_i < \min_i b_i$. Without loss of generality we can assume that this property holds for the instance X of N3DM (else we can increase every $b_i, d_i = 1, 2, \ldots, n$ by $\max_i a_i$ and consider the resulting instance as our X).

To prove the claim, assume that there exists a solution of X such that $a_i + b_i + d_i = \bar{W}$ for every $i = 1, 2, \ldots, n$. Then, for each triplet (a_i, b_i, d_i) form the 8-job block indicated below.

$$
\begin{array}{llllllllll}
24T & 3T+a_i & & & & & & & & \\
 & 21T & 6T+a_i & & & & & & & \\
 & & 18T+b_i & 9T & & & & & & \\
 & & & 15T+\bar{W}-d_i & 12T+\bar{W} & & & & & \\
 & & & & 12T & 15T+\bar{W}-d_i & & & & \\
 & & & & & 9T & 18T+b_i & & & \\
 & & & & & & 6T+a_i & 21T+a_i & & \\
 & & & & & & & 3T & 0 &
\end{array}
$$

Let S be the schedule formed by concatenating the 8-job blocks (n of them) in any order, and process job $(24T, 24T)$ last. Let (a_1, b_1, d_1) be the triplet corresponding to the first 8-job block of S. Observe that the workforce requirements in S for the first 8 production cycles are $W_1 = 24T$, $W_2 = W_8 = 24T + a_1$, $W_3 = W_7 = 24T + a_1 + b_1$, $W_4 = W_6 = 24T + \bar{W} - d_1$, $W_5 = 24T + \bar{W}$. Therefore, the contribution of the first block of 8 jobs to the objective $\sum_k \Delta_k$ is $\sum_{k=1}^{8} \Delta_{k,k+1} = 2(a_1 + b_1 + d_1)$. Similarly, the contribution of the i-th 8-job block is $2(a_i + b_i + d_i)$, and hence $\sum_{k=1}^{8n+1} \Delta_{k,k+1} = 2\sum_i (a_i + b_i + d_i) = 2n\bar{W}$ because $a_i + b_i + d_i = \bar{W}$ for every $i = 1, 2, \ldots, n$. This proves that the claim is a sufficient condition.

To show that it is also necessary, suppose that S is a schedule for $2TL//\sum_k \Delta_k$ such that $\sum_k \Delta_{k,k+1} = 2n\bar{W}$. We can show that any two consecutive jobs of S, say (W_{i1}, W_{i2}) and (W_{j1}, W_{j2}), should be such that $24T \leqslant W_{i2} + W_{j1} < 25T$. To see this, note that the total workload required by all jobs is

$$\sum_{ij} W_{ij} = 192nT + n\bar{W} + \sum_{i=1}^{n} (6a_i + 4b_i) < (192n + 1)T$$

because $\sum_{i=1}^{n} (6a_i + 4b_i) < 4\sum_i (a_i + b_i + d_i) = 4n\bar{W}$ (since $a_i < d_i$ for every $i = 1, 2, \ldots, n$), and $T = 5n\bar{W}$. The above expression yields that $192nT \leqslant \sum_{ij} W_{ij} < (192n + 1)T$. Since the workforce requirements $\sum_{ij} W_{ij}$ are over $8n$ production cycles, the workload during every cycle should be between $24T$ and $25T$ and job $(24T, 25T)$ must be scheduled last, otherwise there will be two adjacent production cycles c_k and c_{k+1} for which $\Delta_{k,k+1} \geqslant T - \bar{W} = (5n-1)\bar{W} > 2n\bar{W}$. This contradicts our assumption that $\sum_k \Delta_{k,k+1} = 2n\bar{W}$ in schedule S. Therefore, in S we must have $24T \leqslant W_k < 25T$ for every $k = 1, 2, \ldots, 8n+2$. This means that S is formed by concatenating n blocks of the form depicted below, with each block consisting of 8-jobs. Consequently, job $(24T, 24T)$ must be scheduled last in S.

$24T$	$3T+a_i$								
	$21T$	$6T+a_j$							
		$18T+b_k$	$9T$						
			$15T+\bar{W}-d_l$	$12T+\bar{W}$					
				$12T$	$15T+\bar{W}-d_r$				
					$9T$	$18T+b_s$			
						$6T+a_u$	$21T+a_u$		
							$3T$	0	

Suppose that the block depicted above is the i-th in the sequence S $(1 \leqslant i \leqslant n)$. Observe that $\Delta_{1+8i,2+8i} = a_i$, $\Delta_{2+8i,3+8i} = a_j + b_k - a_i$, $\Delta_{3+8i,4+8i} = |\bar{W} - d_l - b_k - a_j|$, $\Delta_{4+8i,5+8i} = d_l$, $\Delta_{5+8i,6+8i} = d_r$, $\Delta_{6+8i,7+8i} = |\bar{W} - d_r - b_s - a_u|$, $\Delta_{7+8i,8+8i} = b_s$, $\Delta_{8+8i,9+8i} = a_u$. Then, we have that

$$\sum_{k=1}^{8n+1} \Delta_{k,k+1} = 2\sum_i (a_i + b_i + d_i) + \sum |\bar{W} - d_l - b_k - a_j| + \sum |\bar{W} - d_r - b_s - a_u|.$$

Since $2\sum_i (a_i + b_i + d_i) = 2n\bar{W}$, we have that $\sum_k \Delta_{k,k+1} = 2n\bar{W}$ in S if and only if $\bar{W} = a_j + b_k + d_l$ and $\bar{W} = a_u + b_s + d_r$ in every one of the n blocks of jobs. This means that the triplets (a_j, b_k, d_l) corresponding to the first 4 of the 8 jobs in each block determine a solution of X for the N3DM problem. Similarly, the triplets (a_u, b_s, d_r) determine another solution for instance X. This proves that the claim is a necessary condition as well. This completes the proof of the theorem. □

COROLLARY 3. *The problem* $2TL/cyclic/\sum_k \Delta_{k,k+1}$ *is* $\mathcal{NP}$*-complete in the strong sense.*

Proof. The instance I used in this proof is the same as in Theorem 3 except that job $(24T, 24T)$ is replaced by $(24T, 0)$. With this change, the complexity proof of Theorem 3 carries through for the cyclic case; the zero operation of $(24T, 0)$ ensures that $W_1 = W_{8n+1} = 24T$ in the cyclic schedule. This completes the proof of the corollary. □

The above complexity results indicate that workforce planning with the leveling objectives $\Delta_{\max}$ or $\sum_k \Delta_k$ is very difficult even for $m = 2$ stations. In the next section we show that $2TL//R_{\max}$ is $\mathcal{NP}$-complete in the strong sense. This result requires analysis of a related graph theory problem which we refer to as the *Complementary Hamiltonian Cycle* problem, or CHC.

4. Problem 2TL//R$_{\max}$ is NP-Complete

Consider the following decision problem: Does there exist a cyclic permutation of the jobs in J such that $LB \leqslant W_k \leqslant UB$ for $1 \leqslant k \leqslant n$? If the answer is yes, then we have $R^\star \leqslant UB - LB$. If we have a procedure that solves this decision problem, then we can search over all potential values of LB and UB (which are of polynomial order) and thus identify the optimal value $R^\star$. For given values LB, UB we can cast the decision problem as a Hamiltonian cycle problem on a bipartite graph $\mathcal{B}(LB, UB)$ as follows.

Consider the complete bipartite graph $B(X, Y)$ where the nodes in X, Y represent the n tasks in stations 1 and 2 respectively. Associated to every $x_i \in X$ is the label W_{i1}, and to each $y_i \in Y$ the label W_{i2} for $i = 1, 2, \ldots, n$. The weight of the arbitrary edge $e_{ij} = (x_i, y_j)$ of $B(X, Y)$ is $w_{ij} = W_{i1} + W_{j2}$. Given the graph B and a trial range (LB, UB) of workforce requirements, we denote by $\mathcal{B}(LB, UB)$ the subgraph that includes only edges $e_{ij} \in E(B)$ for which $LB \leqslant w_{ij} \leqslant UB$.

The edges $M = \{(W_{i1}, W_{i2}) : \ i = 1, 2, \ldots, n\}$ of $E(B)$ form a matching in $\mathcal{B}(LB, UB)$. This matching corresponds to the jobs in J and hence it is referred to as the *job matching*. It is easy to observe that every cyclic permutation of the n jobs in J with $LB \leqslant W_k \leqslant UB$ for $k = 1, 2, \ldots, n$, corresponds to an alternating Hamiltonian cycle of $\mathcal{B}(LB, UB)$ whose edges alternate between edges in M and edges in $E(\mathcal{B}(LB, UB))$. For given graph $\mathcal{B}(LB, UB)$ and matching M, we refer to the problem of finding an alternating Hamiltonian cycle as the *Complementary Hamiltonian Cycle* problem, or CHC.

In what follows we show that the decision problem $2TL/cyclic/LB \leqslant W_k \leqslant UB$ is NP-complete. This would mean that $2TL/cyclic/R_{\max}$ is $\mathcal{NP}$-complete as well. We do this by showing that the CHC problem in $\mathcal{B}(LB, UB)$ is NP-complete. For our NP-completeness proof we use a reduction from the Hamiltonian-cycle problem on a cubic graph (that is, a graph in which every node has degree 3). Since every instance of $2TL/cyclic/LB \leqslant W_k \leqslant UB$ can be cast as a CHC problem in the bipartite graph $\mathcal{B}(LB, UB)$, it is simpler to first construct a bipartite graph $\mathcal{B}(LB, UB)$ and then specify the associated instance of $2TL/cyclic/LB \leqslant W_k \leqslant UB$ represented by it. This is what we do next.

Consider a cubic graph G with node set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and arc set $E(G)$. Given G, construct the bipartite graph $\mathcal{B}(X, Y)$ as follows. Associated with each $v_i \in V(G)$ is a bipartite subgraph B_i consisting of 7 pairs of nodes, as shown in Figure 1. Hence, the node set of $\mathcal{B}(X, Y)$ consists of $14n$ nodes, and $|X| =$

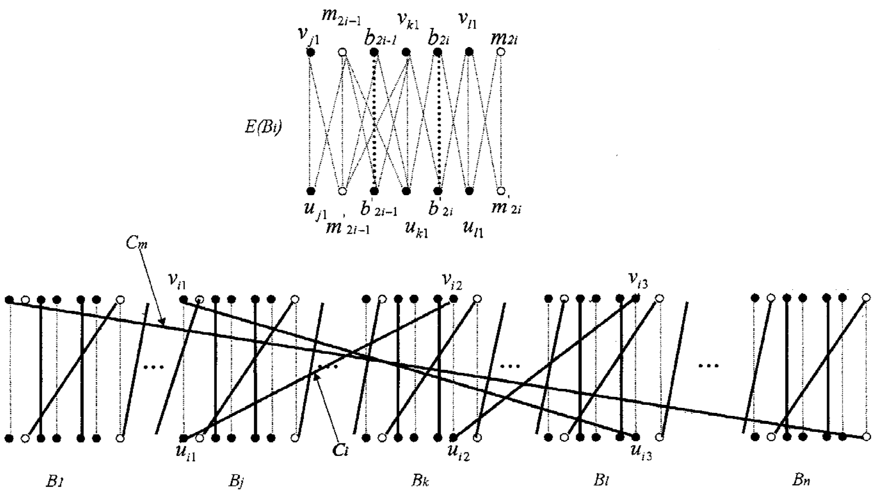

Figure 1. The subgraph B_i, the edge set $E(B_i)$, and the matching M

$|Y| = 7n$. As shown in Figure 1,

$$X = \{v_{i1}, v_{i2}, v_{i3} \ : \ 1 \leqslant i \leqslant n\} \cup \{m_{2i-1}, m_{2i} \ : \ 1 \leqslant i \leqslant n\} \cup \{b_{2i-1}, b_{2i} \ : \ 1 \leqslant i \leqslant n\}$$

and

$$Y = \{u_{i1}, u_{i2}, u_{i3} \ : \ 1 \leqslant i \leqslant n\} \cup \{m'_{2i-1}, m'_{2i} \ : \ 1 \leqslant i \leqslant n\} \cup \{b'_{2i-1}, b'_{2i} \ : \ 1 \leqslant i \leqslant n\}.$$

The nodes $m_{2i-1}, m_{2i}, m'_{2i-1}, m'_{2i}$ are referred to as “master” nodes because, as is shown shortly, they are used to form a “master” subcycle. Similarly, the nodes $v_{i1}, v_{i2}, v_{i3}, u_{i1}, u_{i2}, u_{i3}$ for $1 \leqslant i \leqslant n$ are referred to as V-nodes because they are going to be included in subcycles corresponding to vertices in $V(G)$. The reason for introducing 3 pairs of nodes (v_{i1}, u_{i1}), (v_{i2}, u_{i2}), (v_{i3}, u_{i3}) for each $v_i \in V(G)$ is that, in our construction, they provide a way to traverse different subgraphs B_k; namely, the 3 subgraphs that contain the pairs (v_{i1}, u_{i1}), (v_{i2}, u_{i2}) and (v_{i3}, u_{i3}). Finally, nodes $b_{2i-1}, b_{2i}, b'_{2i-1}, b'_{2i}$ for $1 \leqslant i \leqslant n$ are referred to as b-nodes and are used to form the matching M of the complementary Hamiltonian cycle problem.

The V-nodes of each B_i correspond to the 3 vertices incident to v_i in the cubic graph G. Hence, if the nodes linked with v_i in G are v_j, v_k and v_l, the V-nodes of B_i will be (in our construction) $v_{j1}, u_{j1}, v_{k1}, u_{k1}, v_{l1}, u_{l1}$; see Figure 1. Since every $v_i \in V(G)$ is linked to precisely 3 vertices, say v_j, v_k and v_l, precisely one of the V-pairs (v_{i1}, u_{i1}), (v_{i2}, u_{i2}), or (v_{i3}, u_{i3}) will appear in each of the subgraphs B_j, B_k and B_l. Evidently, pairs (v_{i1}, u_{i1}), (v_{i2}, u_{i2}), and (v_{i3}, u_{i3}) are used in our construction as a vehicle to traverse subgraphs B_j, B_k and B_l. Also, observe that each B_i includes 3 pairs of V-nodes corresponding to the 3 nodes in $V(G)$ linked with v_i in G. The remaining nodes of B_i are 2 pairs of b-nodes and 2 pairs of master nodes.

To complete the construction of $\mathcal{B}$(X,Y) it remains to describe the edge set $E(\mathcal{B})$. Let $E(\mathcal{B}) = \cup_{i=1}^{n} E(B_i)$ where $E(B_i)$ consists of all dotted edges depicted in Figure 1. Evidently, no edge in $E(\mathcal{B})$ has its endpoints in different B_i's. To completely describe an instance of the CHC problem in $\mathcal{B}(X, Y)$ we also need to describe a job matching M of $\mathcal{B}(X, Y)$. Since $E(\mathcal{B}(X, Y))$ does not include edges with endpoints in different B_i subgraphs, the choice of matching M is crucial for the existence or not of a CHC in $\mathcal{B}(X, Y)$. Before we introduce M we need the following notation. Let us denote edges in M by $[v_i, u_j]$ and edges in $E(\mathcal{B}(X, Y))$ by (v_i, u_j). For brevity, we use $v_i - u_j$ to denote $[v_i, u_j]$ and $v_i \dots u_j$ to denote (v_i, u_j). An alternating sequence $[x_1, y_1'], (y_1', x_2), [x_2, y_2'], (y_2', x_3) \dots$ of edges will be denoted as $x_1 - y_1' \dots x_2 - y_2' \dots x_3 - \dots$. We now proceed with the description of M.

Let $M = M_1 \cup M_2 \cup M_3$, where

$$M_1 = \{(b_{2i-1}, b_{2i-1}'), (b_{2i}, b_{2i}') \ : \ 1 \leqslant i \leqslant n\}.$$

That is, M_1 matches the 2 pairs of b-nodes in each B_i and $b_k - b_k' \dots b_k$ form trivial subcycles for $1 \leqslant k \leqslant 2n$. Let

$$M_2 = \{((u_{i1}, v_{i2}), (u_{i2}, v_{i3}), (u_{i3}, v_{i1}) \ : \ 1 \leqslant i \leqslant n\}.$$

That is, M_2 matches the 6 V-nodes associated with each $v_i \in V(G)$ in such a way that, for every $1 \leqslant i \leqslant n$, $C_i = u_{i1} - v_{i2} \dots u_{i2} - v_{i3} \dots u_{i3} - v_{i1} \dots u_{i1}$ is a cycle that traverses the 6 V-nodes associated with each $v_i \in V(G)$. Finally, let

$$M_3 = \{(m_1', m_2), (m_2', m_3), (m_3', m_4), \dots, (m_{2n-1}', m_{2n}), (m_{2n}', m_1)\}.$$

That is, M_3 matches in pairs all master nodes in such a way that $C_m = m_1' - m_2 \dots m_2' - m_3 \dots m_3' - m_4 \ \dots \ m_{2n-1}' - m_{2n} \dots m_{2n}' - m_1 \dots m_1'$ forms a single cycle traversing all master nodes. By construction of M and $E(\mathcal{B}(X, Y)$, C_m is the only subcycle visiting all of the subgraphs B_i, $1 \leqslant i \leqslant n$.

In order to find a complementary Hamiltonian cycle in $\mathcal{B}(X, Y)$ for the matching M, it is necessary to find matchings $I_1, I_2, \dots, I_n$ where $I_i \subset E(B_i)$ for $1 \leqslant i \leqslant n$ so that $\cup_{i=1}^{n} I_i \cup M$ forms a unique cycle that merges together the trivial subcycles $b_k - b_k' \dots b_k$ for $1 \leqslant k \leqslant 2n$, the cycles $C_i \ : \ 1 \leqslant i \leqslant n$, and the master cycle C_m. For the particular graph $\mathcal{B}(X, Y)$ and the matching M described above, it is shown in [7] that there exists a complementary Hamiltonian cycle for $\mathcal{B}(X, Y)$ if and only if there exists a Hamiltonian cycle for the cubic graph G. In the following theorem we provide an instance of $2TL/cyclic/LB \leqslant W_k \leqslant UB$ for which the bipartite graph $\mathcal{B}(LB, UB)$ coincides with $\mathcal{B}(X, Y)$.

THEOREM 4. *Problem $2TL/cyclic/R_{\max}$ is $\mathcal{NP}$-complete in the strong sense.*

Proof. Consider any integer $K \geqslant 7$, and the threshold values $LB = (n+1)K-1$ and $UB = (n+1)K + 1$. We will construct an instance of $2TL/cyclic/LB \leqslant$

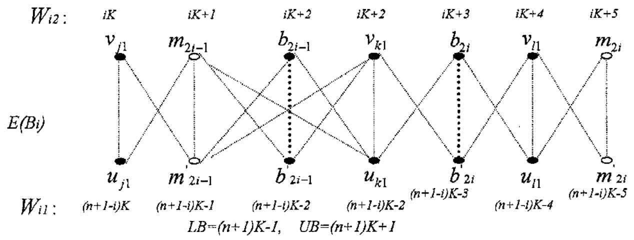

Figure 2. The workforce requirements for the subgraph B_i, of $B(LB, UB)$

$W_k \leqslant UB$ for which $\mathcal{B}(LB, UB) = \mathcal{B}(X, Y)$. Evidently, our instance will consist of $7n$ jobs. To describe the workforce requirements (W_{i2}, W_{i1}) of each each job J_i it is enough to associate values W_{i2} and W_{i1} with each $v_i \in X$ and $u_i \in Y$ respectively. Such pairs are provided in Figure 2 for the subgraph B_i, $i = 1, 2, \ldots, n$. Given these values, we have to verify that $\mathcal{B}(X, Y) = \mathcal{B}(LB, UB)$ for $2TL/cyclic/LB \leqslant W_k \leqslant UB$. Equivalently, we need to show that the edge set $E(\mathcal{B}(X, Y))$ coincides with $E(\mathcal{B}(LB, UB))$.

Indeed, observe that by choice of the LB, UB values, in $\mathcal{B}(LB, UB)$ no edge (u_x, v_y) exists with $u_x \in B_i$ and $v_y \in B_l$ with $1 \leqslant i \neq l \leqslant n$. This is because by definition of B_i, the number of workers associated with $v_x \in B_i$ is $W_{x1} = (n-i+1)K - w_1$ where $0 \leqslant w_1 \leqslant 5$. Similarly, the number of workers associated with $v_y \in B_l$ is $W_{y2} = lK + w_2$ where $0 \leqslant w_2 \leqslant 5$. Therefore, the workforce requirement $w_{x,y}$ associated with the edge (u_x, v_y) in $\mathcal{B}(LB, UB)$ is

$$\begin{aligned} w_{xy} &= W_{x1} + W_{u2} = [(n-i+1)K - w_1] + [lK + w_2] \\ &= (n+1)K + (l-i)K + (w_2 - w_1). \end{aligned}$$

If $i > l$, we have

$$w_{xy} \leqslant (n+1)K - K + w_2 < (n+1)K - 1 = LB$$

because $K \geqslant 7$ and $w_2 \leqslant 5$. On the other hand, if $i < l$, we have

$$w_{xy} \geqslant (n+1)K + K - w_1 > (n+1)K + 1 = UB$$

because $K \geqslant 7$ and $w_1 \leqslant 5$. Hence, $E(\mathcal{B}(LB, UB))$ does not include edges connecting nodes of different B_i subgraphs. In addition, it easy to verify that, within each B_i, the edges (v_i, u_l) depicted in Figure 2 are the only edges for which the required number of workers is less than or equal to $UB = (n+1)K+1$ and greater or equal to $LB = (n+1)K - 1$.

Therefore $\mathcal{B}(X, Y) = \mathcal{B}(LB, UB)$ which means that there exists a solution for $2TL/cyclic/LB \leqslant W_k \leqslant UB$ if and only if there exists a CHC in $\mathcal{B}(X, Y)$ for the given matching M. As shown in [7] there exists a CHC for $\mathcal{B}(X, Y)$ if and only if there exists a Hamiltonian cycle for the cubic graph G. The latter problem is known to be strongly NP-complete; see Garey and Johnson, 1979. Hence,

$2TL/cyclic/LB \leqslant W_k \leqslant UB$ is strongly NP-complete and so is $2TL/cyclic/R_{\max}$. This completes the proof of the theorem. □

COROLLARY 4. *The problem* $2TL//R_{\max}$ *is* $\mathcal{NP}$*-complete in the strong sense.*

Proof. Consider the same instance as in Theorem 4 except for arc $[b_1, b_1'] \in M$. Instead, introduce two new edges: $[v, b_1]$ and $[b_1', u]$. Still, the workforce requirements associated with b_1, b_1' are $W_{b_1,2} = K + 2$ and $W_{b_1',1} = nK - 2$ respectively. Hence, the edges of $E(B_1)$ incident to nodes b_1 and b_1' remain the same even after replacing $[b_1, b_1']$. Let the workforce requirements associated with the new nodes be $W_{v,1} = LB$ and $W_{u,2} = UB$ respectively. The only new edge added to $E(\mathcal{B}(LB, UB))$ is edge (u, v). Therefore, the only edge in $\mathcal{B}(LB, UB)$ available to link $[b_1', u]$ and $[v, b_1]$ is (u, v). Hence, if there exists a CHC for $\mathcal{B}(LB, UB)$ with respect to $M - [b_1, b_1'] + [v, b_1] + [b_1', u]$, it must use the path $[b_1', u]$, (u, v), $[v, b_1]$. Upon finding such CHC, we can convert the resulting cyclic permutation of jobs to a non-cyclic one by executing job $[v, b_1]$ first, and job $[b_1', u]$ last. With this modification, every instance of $2TL/cyclic/LB \leqslant W_k \leqslant UB$ can be converted to an instance of $2TL//LB \leqslant W_k \leqslant UB$ and vice-versa. Then, in light of Theorem 4 problem $2TL//R_{\max}$ is $\mathcal{NP}$-complete in the strong sense. □

5. Conclusions

In this article we defined a variety of objective functions for the workforce planning problem on synchronous production systems and determined the complexity status of the corresponding problems. To the best of our knowledge no other leveling objectives have been presented before in the literature for day-to-day tactical scheduling operations. Our survey of related literature shows that there is very little research done on level workforce measures even though leveling issues are of practical importance in manufacturing settings. We showed that, except for $2TL//W_{min}$ that is solvable in $\mathcal{O}(n \log n)$ time (see [9]), all other problems are strongly NP-complete even for 2-station paced assembly lines. This means that workforce leveling is a very hard problem. Hence, increased effort is required to find reasonable solutions for such problems.

In addition to the basic problems formulated in this article, future research should address cross-training issues in synchronous production systems. In this setting, workers are not trained to work on every single station of the assembly system, but only on a small subset of stations as dictated by the nature of the work. Evidently, this article also provides a complexity classification for many cross-training problems. Namely, if there is a skill with 3 or more stations, then the workforce planning problem is NP-complete for any objective f. Research on cross-training issues is important not only for tactical decision making but also for gaining insight on effective ways to form skill vectors (i.e., determine the stations on which workers of a particular skill are trained) so as to minimize cross-training costs.

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